FURTHER MATH

Summer Assignment 2017

Name: ___________________________________

Due Date: Monday, August 28th

No late assignments will be accepted.

DIRECTIONS

Complete the packet in entirety (be neat and organized). It is due on the first day of classes.

You will be given a grade on completeness/accuracy.

I encourage you to work in groups, however, I do not expect everyone's papers to be identical.

You may reference Get a Five, Khan Academy, YouTube, etc. for tutorials when necessary.

Please list any sources your referenced below.

List of Sources Used (websites, textbooks, etc):

Summer Assignment Grade: __________

PART I (#1 - #12): Short Response
Please answer each part of each problem completely. Please show all work and circle all final answers.

1. (a) Write down and simplify the expansion of $(2+x)^4$ in ascending powers of x.

   (b) Hence find the exact value of $(2.1)^4$.

2. A given polynomial function is defined as $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$. The roots of the polynomial equation $f(x) = 0$ are consecutive terms of a geometric sequence with a common ratio of $\frac{1}{2}$ and first term 2.

   Given that $a_{n-1} = -63$ and $a_n = 16$ find

   (a) the degree of the polynomial;

   (b) the value of a_0.

3. A box contains four red balls and two white balls. Darren and Marty play a game by each taking it in turn to take a ball from the box, without replacement. The first player to take a white ball is the winner.

   (a) Darren plays first, find the probability that he wins.

   The game is now changed so that the ball chosen is replaced after each turn.
   Darren still plays first.

   (b) Show that the probability of Darren winning has not changed.

4. Josie has three ways of getting to school. 30% of the time she travels by car, 20% of the time she rides her bicycle and 50% of the time she walks.
When travelling by car, Josie is late 5% of the time. When riding her bicycle she is late 10% of the time. When walking she is late 25% of the time. Given that she was on time, find the probability that she rides her bicycle.

5. Ed walks in a straight line from point $\mathrm{P}(-1, 4)$ to point $\mathrm{Q}(4, 16)$ with constant speed. Ed starts from point P at time $t = 0$ and arrives at point Q at time $t = 3$, where t is measured in hours.
Given that, at time t, Ed's position vector, relative to the origin, can be given in the form, $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$,

(a) find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Roderick is at a point $\mathrm{C}(11, 9)$. During Ed's walk from P to Q Roderick wishes to signal to Ed. He decides to signal when Ed is at the closest point to C.

(b) Find the time when Roderick signals to Ed.

6. The three planes having Cartesian equations $2x + 3y - z = 11$, $x + 2y + z = 3$ and $5x - y - z = 10$ meet at a point P. Find the coordinates of P.

7. Consider the curve $y = \frac{1}{1-x}$, $x \in \mathbb{R}$, $x \neq 1$.

(a) Find $\frac{dy}{dx}$.

(b) Determine the equation of the normal to the curve at the point $x = 3$ in the form $ax + by + c = 0$ where $a, b, c \in \mathbb{Z}$.

8. A curve is defined by $xy = y^2 + 4$.

(a) Show that there is no point where the tangent to the curve is horizontal.

(b) Find the coordinates of the points where the tangent to the curve is vertical.

9. A particle can move along a straight line from a point O. The velocity v, in $\mathrm{m\,s^{-1}}$, is given by the function $v(t) = 1 - \mathrm{e}^{-\sin t^2}$ where time $t \geq 0$ is measured in seconds.

(a) Write down the first two times $t_1, t_2 > 0$, when the particle changes direction.

(b) (i) Find the time $t < t_2$ when the particle has a maximum velocity.

(ii) Find the time $t < t_2$ when the particle has a minimum velocity.

(c) Find the distance travelled by the particle between times $t = t_1$ and $t = t_2$.

10. Use the substitution $u = \ln x$ to find the value of $\int_{e}^{e^2} \frac{dx}{x \ln x}$.

11. Using integration by parts find $\int x \sin x \, dx$.

12. A function is defined by $f(x) = x^2 + 2$, $x \geq 0$. A region R is enclosed by $y = f(x)$, the y-axis and the line $y = 4$.

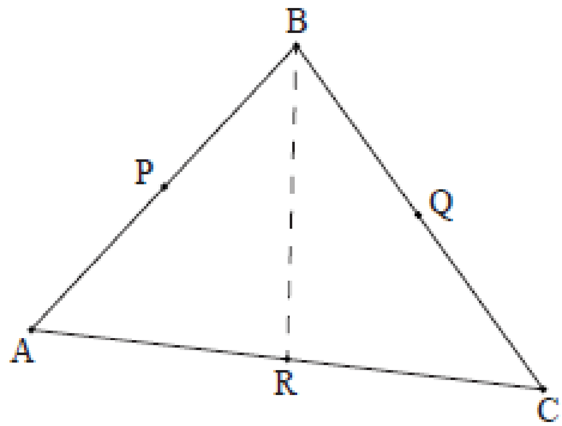
(a) (i) Express the area of the region R as an integral with respect to y.

(ii) Determine the area of R, giving your answer correct to four significant figures.

(b) Find the exact volume generated when the region R is rotated through 2π radians about the y-axis.

PART II (#13 - #17): Extended Response
Please answer each part of each problem completely. Please show all work and circle all final answers.

13.



Consider the triangle ABC. The points P, Q and R are the midpoints of the line segments [AB], [BC] and [AC] respectively.

Let $\vec{OA} = a$, $\vec{OB} = b$ and $\vec{OC} = c$.

(a) Find $\vec{BR}$ in terms of a, b and c.

(b) (i) Find a vector equation of the line that passes through B and R in terms of a, b and c and a parameter λ.

(ii) Find a vector equation of the line that passes through A and Q in terms of a, b and c and a parameter μ.

(iii) Hence show that $\vec{OG} = \frac{1}{3}(a + b + c)$ given that G is the point where [BR] and [AQ] intersect.

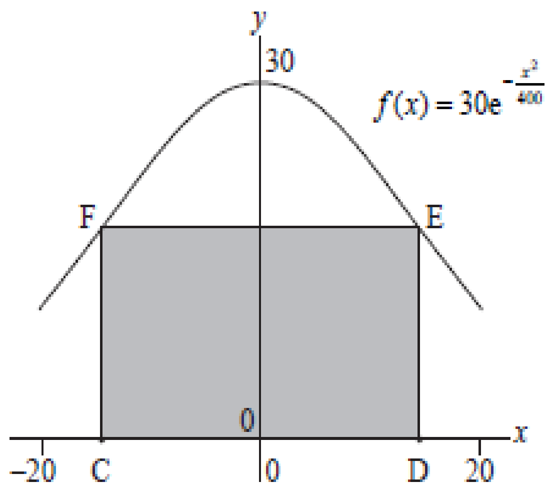
(c) Show that the line segment [CP] also includes the point G.

The coordinates of the points A, B and C are $(1, 3, 1)$, $(3, 7, -5)$ and $(2, 2, 1)$ respectively.
A point X is such that [GX] is perpendicular to the plane ABC.

(d) Given that the tetrahedron ABCX has volume 12 units^3, find possible coordinates of X.

14. The following diagram shows a vertical cross section of a building. The cross section of the roof of the building can be modelled by the curve $f(x) = 30\mathrm{e}^{-\frac{x^2}{400}}$, where $-20 \le x \le 20$.

Ground level is represented by the x-axis.



(a) Find $f''(x)$.

(b) Show that the gradient of the roof function is greatest when $x = -\sqrt{200}$.

The cross section of the living space under the roof can be modelled by a rectangle CDEF with points $\mathrm{C}(-a, 0)$ and $\mathrm{D}(a, 0)$, where $0 < a \le 20$.

(c) Show that the maximum area A of the rectangle CDEF is $600\sqrt{2}\mathrm{e}^{-\frac{1}{2}}$.

(d) A function I is known as the Insulation Factor of CDEF. The function is defined as $I(a) = \dfrac{P(a)}{A(a)}$ where P = Perimeter and A = Area of the rectangle.

(i) Find an expression for P in terms of a.

(ii) Find the value of a which minimizes I.

(iii) Using the value of a found in part (ii) calculate the percentage of the cross sectional area under the whole roof that is not included in the cross section of the living space.

15. Let $f(x) = e^x \sin x$.

(a) Show that $f''(x) = 2(f'(x) - f(x))$.

(b) By further differentiation of the result in part (a), find the Maclaurin expansion of $f(x)$, as far as the term in x^5.

16. The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1, 0)$ and are defined by the differential equations $\frac{dy}{dx} = x - y^2$ and $\frac{dy}{dx} = y - x^2$ respectively.

(a) Show that the tangent to the curve $y = f(x)$ at the point $(1, 0)$ is normal to the curve $y = g(x)$ at the point $(1, 0)$.

(b) Find $g(x)$.

(c) Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places.

17. (a) Prove by induction that $n! > 3^n$, for $n \geq 7$, $n \in \mathbb{Z}$.

(b) Hence use the comparison test to prove that the series $\sum_{r=1}^{\infty} \frac{2^r}{r!}$ converges.